Incompressible Hydrodynamics on a Noncommutative D3-Brane

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Abstract

In the Seiberg-Witten limit, the low-energy dynamics of N weakly coupled identical open strings on a D3-brane can behave as two-dimensional incompressible hydrodynamics. Classical vortices are frozen in the fluid and described by an action expressed in terms of two canonical conjugate fields, which can be taken as the new space coordinate. The noncommutative space naturally arises when this pair of conjugate fields are quantized. To the lowest order of $\hbar$, the vorticity can replace the background B-field on the D3 brane, thereby yielding a spatially and temporally varying noncommutative parameter θ^{ij}. Demanding a quantum area-preserving transformation between two classical inertial-frame coordinates, we identify the classical solitons that survive in the noncommutative space, and they turn out to be the "electric field" solutions of the Dirac-Born-Infeld Lagrangian created by a δ-function source. The strongly magnetized electron gas in a semiconductor of finite thickness is taken as a case study, where similar quantum column vortices as those in a D3 brane can be present. The electric charges contained in these electron-gas column vortices are quantized, but in a way different from those in the sheet vortices that produce the fractional quantum Hall effect.

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I. INTRODUCTION

Strings can have endpoints, which anchor on the D-branes. Such open strings have interesting connections with the Yang-Mills theory when there exists an antisymmetric tensor field $B_{\mu\nu}$ on the D-branes [1–8]. Normally, the antisymmetric field is magnetic-like; that is, the string endpoints do not experience the electric field $E_i = B_{0i}$. Moreover, due to the presence of the background field on the D-brane, the dynamics of the string endpoints is different from that of the string interior. The disparity of dynamics can be built in their kinematics by different metrics [9,10]. The closed-string metric $g_{\mu\nu}$ describes that of the original space. The open-string metric $G_{\mu\nu}$ is an effective one, describing the oscillating motion of string endpoint on the D-brane due to coupling to the oscillation modes in the string bulk.

The effective metric $G_{\mu\nu}$ is related to $g_{\mu\nu}$ via the field $B_{\mu\nu}$ by the relation

$$G_{\mu\nu} = g_{\mu\nu} - (\alpha')^2 B_{\mu\xi}(g^{-1})^{\xi\eta} B_{\eta\nu}, \tag{1}$$

where $(\alpha')^{-1}$ is the string tension, and raising or lowering indices always refers to the original metric $g_{\mu\nu}$ and indices run from 0 to p for a p-dimensional brane, the Dp-brane. Equation (1) shows that only the components of $B_{\mu\nu}$ that are parallel to the D-brane matter, and hence other components can be set to zero.

The following simple picture, which is generalized to include an "electric" field, can be helpful for understanding the basic dynamics of an open string on a D2-brane. For notational simplicity, we let $B_z \equiv B_{12}$, to be conceived as the magnetic field, and $E_x \equiv B_{01}$, $E_y \equiv B_{02}$ as the electric field. The string bulk does not interact with a uniform background field, and only the endpoint does. We thus let a charge be attached to the string endpoint, which moves on the $x-y$ plane perpendicular to the magnetic field. Apart from the electromagnetic forces, the endpoint also experiences a tension force, $\mathbf{T}(x, y)$, parallel to the D2-brane, as the string is generally not perpendicular to the brane. The string endpoint has zero inertia, and the combined electric and tension forces, $\mathbf{F} = q\mathbf{E} + \mathbf{T}(x, y)$, must instantaneously be balanced by a magnetic force $q\dot{\mathbf{x}} \times B\hat{z}$. It follows that the endpoint velocity $\dot{\mathbf{x}} = \mathbf{F} \times \hat{z}/qB$. As the velocity is always perpendicular to the force, there is no energy exchange between the string endpoint and the electromagnetic/tension fields. The physical origin of this peculiar feature is that the string endpoint has zero inertia, thus containing no energy.

In mathematical terms, the endpoint equation of motion reads:

$$B_{\mu\nu}\partial_t x^\nu = -(\alpha')^{-1} g_{\mu\nu}\partial_n x^\nu, \tag{2}$$

where n is the unit direction along the string and t is the time. The electromagnetic forces are on the left and the tension force is on the right.

The endpoint can oscillate in response to the oscillation modes within the string bulk. A Green's function calculation [11–13] shows that the auto-correlation function of the endpoint motion reads

$$\langle x^\mu(t)x^\nu(0)\rangle = i\alpha'(G^{-1})^{\mu\nu}\log t^2 + (i/2)\theta^{\mu\nu}\epsilon(t), \tag{3}$$

where $\epsilon(t)$ equals 1 or -1 for positive or negative time t, and the anti-symmetric tensor $\theta^{\mu\nu}$ is given as

$$\theta^{\mu\nu} \equiv 2\pi(\alpha')^2(g^{-1}BG^{-1})^{\mu\nu}. \tag{4}$$

The first term of the auto-correlation function represents the time symmetric coupling to the bulk oscillations where the open-string metrics $G_{\mu\nu}$ plays the role of an effective metric; the second term carries the time direction, representing a net displacement around a circle acquired from the secular drift (zero-frequency mode) of endpoint motion. The above Green's function is obtained in the frame where the "electric" field B_{0i} is absent.

When the string endpoint is attached to a D3-brane, a situation relevant to the present work, one can think of the charged particle to be in a three-dimensional (3D) space in the presence of a magnetic field lying within the 3D brane world volume. The magnetic field lines break the isotropy of the 3D space into two transverse directions and one longitudinal direction. The string extends out of the 3D space into the higher dimensions, with its tension force oriented to an arbitrary direction in the 3D space. In the direction perpendicular to the magnetic field line, the force balance is identical to that on a D2-brane described above. But the component of tension force along the magnetic field line is now unbalanced, causing the endpoint to oscillate freely along the field line in response to the bulk oscillations.

On a D3-brane, the Seiberg-Witten map [9] takes the decoupling limit, $g_{00} = -g_{33} = -1$, $g_{ij} \to \epsilon^2 \delta_{ij}$ and $\alpha' \to \epsilon$, with $\epsilon \to 0$ and i, j running from 1 to 2. A straightforward calculation shows that

$$\theta^{ij} \to 2\pi (B^{-1})^{ij} = 2\pi B^{-1} \epsilon^{ij}, \tag{5}$$

and $\theta^{\mu\nu} \to 0$, otherwise. In the same limit, we also have

$$G_{ij} \to \frac{\alpha'^2 B^2}{\epsilon^2} \delta_{ij}, \quad G_{00} = -1 + \frac{\alpha'^2 B^2 v_j v^j}{\epsilon^2 c^2}, \quad G_{0i} = G_{i0} = \frac{\alpha'^2 B^2 v^i}{\epsilon^2 c}, \tag{6}$$

where $\mathbf{v}/c \to \mathbf{E} \times \hat{z}/B$, the endpoint velocity given in (2) in this decoupling limit. The open-string metric $G_{\mu\nu}$ and θ^{ij} are both finite. The string now becomes a rigid wire, where the string oscillations are hard to excite, according to (3), and to influence the endpoint motion. The net displacement around a circle in (3) remains finite in this limit.

In the presence of a constant electric field $\mathbf{E}$, the endpoint moves at a constant velocity, and this electric field can be transformed away by choosing an appropriate reference frame. It is in this frame that the net displacement of endpoint motion in (3) is calculated, and the quantum field theory on the D-brane becomes such that the D-brane coordinates are noncommutative. That is, the coordinates on the brane are operators that satisfy [4]

$$[x^i, x^j] = i\theta^{ij}. \tag{7}$$

Physically, what happens here is that the momentum, $p_j = -i\hbar\partial_j + eA_j$, is dominated by the "diamagnetic" current, $eA_j(x)$, in the presence of a strong field. By the relation, $[eA_j(x), x_k] \to [p_j, x_k] = -i\hbar\delta_{jk}$, the open-string space becomes noncommutative.

In view of the complications arising from the operator algebras, one may alternatively represent functions of noncommutative coordinates by functions of commutative coordinates, except that the product of any two of the ordinary functions, e.g., f and h, is replaced by the star-product [14], defined as

$$f * h = fh + \frac{i}{2}\theta^{ij}(\partial_i f)(\partial_j h) - \frac{1}{8}\theta^{kl}\theta^{ij}(\partial_k\partial_i f)(\partial_l\partial_j h)$$
$$-\frac{1}{12}\theta^{kl}\partial_k\theta^{ij}((\partial_i f)(\partial_l\partial_j h) - (\partial_l\partial_i f)(\partial_j h)) + O(\theta^3), \tag{8}$$

where all products on the right are ordinary products. The commutator now is $[f, h] \equiv f * h - h * f$. Substituting x_i and x_j for f and h, one recovers (7). It is straightforward to show, by the symmetry of indices, that $[f, h] = i\theta^{ij}(\partial_i f)(\partial_j h) + O(\theta^3)$, where the contribution from the second-order, $O(\theta^2)$, terms vanishes.

Note that the coefficient, 1/12, of the term with $\partial_k\theta^{ij}$ in (8) is chosen such that the star product, to the order $O(\theta^2)$, obeys the associativity [14], $(f * q) * h = f * (q * h)$. Apart from an appropriate coefficient, the associativity further requires

$$H_{ijk} \equiv \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} = 0, \tag{9}$$

or $\nabla \cdot \mathbf{B} = 0$ on the D3-brane. Apparently, a magnetic-like field defined as $B_{ij} \equiv \partial_i A_j - \partial_j A_i$ does satisfy (5). The antisymmetric field $B_{\mu\nu}$ normally is a linear combination of the Neveu-Schwarz (NS) field and a $U(1)$ gauge field. In the present work, we shall assume that the NS field dominantly contributes to $B_{\mu\nu}$.

In the reference frame void of the electric-like NS field, the energy density of the NS field is proportional to $H_{ijk}H^{ijk}$ and hence any NS field that makes $H_{ijk} = 0$ contains no energy. In a general reference frame, the time component should be included and a vacuum NS field obeys

$$H_{\mu\nu\eta} \equiv \partial_\mu B_{\nu\eta} + \partial_\nu B_{\eta\mu} + \partial_\eta B_{\mu\nu} = 0, \tag{10}$$

where the indices include 0 and the additional constraint from (10) on the D3-brane is the Faraday's law. Again, as the energy denisty of the NS field is proportional to $H_{\mu\nu\eta}H^{\mu\nu\eta}$, the field energy is zero.

Equation (10) implies that $B(x, y, t)$ is not only non-uniform but can also be time-dependent. However, the solution to (10) is not well-defined, since we do not know what B_{0i}

should be. In light of the discussions made prior to (7), we demand that the "electric" field B_{0i} vanishes in the rest frame of the string endpoint, i.e., $B'_{0i} = 0$. It provides a condition to relate the "electric" field to the endpoint motion by $E_i \equiv B_{0i} = B_{ij}v^j/c$. This is the perfect conductor limit, which permits the endpoint to move across the "magnetic" field $\mathbf{B}$. The NS field so constructed remains a vacuum field and contain no energy; such an NS field differs from the $U(1)$ gauge field since it lacks the other half of the Maxwell equations. In this paper, we shall identify B_{12} to be the vorticity of the incompressible fluid. Fluid motion contains the kinetic energy of particles, but the presence of vorticity does not add anything extra.

The above relation for v_j and E_i is still somewhat ambiguous in the sense that one may regard either the velocity to be generated by a given "electric" field or the "electric" field generated by a given velocity. In the context of open strings, we may adopt either view. When there exist many open strings on the same D-brane, strings can interact. One situation where such an interaction is inevitable is when the two endpoints of a string are on the same D-brane; the small tension force given by (2) couples them. On the other hand, charged string endpoints can also create a $U(1)$ Coulomb electric field so as to couple to each other; the energy of such a Coulomb field can be made negligibly small compared with the kinetic energy when the denisty of endpoints is sufficiently large, as will be addressed in more details in Sec.(IV).

Indeed it is well known that the guiding-center motion of a strongly magnetized 3D dense electron gas is incompressible and involves only 2D displacements [15]. Their motion is advanced by a self-induced Coulomb electric field. Such an electron-gas system can be the classical counterpart of N weakly coupled open strings on a D3-brane with noncommutative coordinates. We are partially so motivated to focus on the D3-brane, with an emphasis on the non- uniform θ^{ij} in the decoupling limit. Since $B^k(\epsilon^{kij}B_{ij})$ is along one direction on the D3-brane, we let it be $\mathbf{B} = B\hat{z}$. The dynamics of string endpoint in the z direction is almost a free motion in the decoupling limit subject only to the open-string interactions, and in this limit the dynamics in z direction is decoupled from that perpendicular to $\mathbf{B}$, provided that the interactions of open strings are relatively weak. Thus, the relevant dynamics on a D3-brane in the decoupling limit is essentially *two dimensional.*

This paper is organized as follows. Sec.(II) advances the connection of two-dimensional incompressible fluid to the noncommutative space, where the action of incompressible hydrodynamics is obtained, thereby allowing one to identify the suitable canonical coordinate for dealing with space quantization in the presence of a non-uniform and time-dependent θ^{ij}. We then make use of this new coordinate to examine canonical transformations in Sec.(III) and obtain some special classical soliton solutions, which survive in the noncommutative space and which turn out to be the solutions to the Dirac-Born-Infeld Lagrangian. In Sec.(IV), the strongly magnetized electron-gas column is studied as an example of quantum incompressible hydrodynamics. The quantized charge contained in the vortex is quantitatively predicted. Discussions and conclusions are given in Sec.(V).

For clarity of presentation, we shall focus on the low-energy physics, where the non-relativistic theory applies and the incompressibility condition, central to the present discussions, has no ambiguity. (In fact, the decoupling limit also leads to the non-relativistic limit.) Only the decoupling limit in a flat D3-brane, where $g_{00} = -g_{zz} = -1$ and $g_{ij} = \epsilon^2\delta_{ij}$ with i, j being the indices for either x or y coordinates, will be considered here. Furthermore, we conceive that the two endpoints of the rigid string anchor almost perpendicularly onto two neighboring parallel D3-branes, and that the NS field $B_{\mu\nu}$ is inhomogeneous only in the two relevant space coordinates x and y; in this setup, the bulk string velocity is the same as those of the endpoints, perpendicular to the string space coordinate σ, even in the presence of an inhomogeneous NS field in the bulk.

II. ACTION FOR INCOMPRESSIBLE HYDRODYNAMICS

In the conventional notion of a uniform θ^{ij}, one may conceive the commutation relation (7) to be analogous to that of $[x_i, p_j] = i\hbar\delta_{ij}$. The latter quantizes the two-dimensional phase space into many cells of arbitrary shape but of same area. The noncommutative coordinates also quantize the space perpendicular to $\mathbf{B}$ into many unit cells of arbitrary shape. The quantization of phase space is possible only because the phase space in classical mechanics is incompressible. Similarly, the classical fields that precede the quantum fields of noncommutative geometry should also involve incompressible motion in directions perpendicular to $\mathbf{B}$. In a D3-brane, such classical motion of identical particles or identical string endpoints satisfies $\nabla_\perp \cdot \mathbf{v}$=0, where $\perp$ refers to the direction perpendicular to $\mathbf{B}$.

When θ^{12} is non-uniform and time-dependent, the quantized two-dimensional spatial cells on the D3-brane will have different sizes at different locations and time. In contrast to the phase-space quantization, a spatio-temporal varying θ^{12} seems to make the quantization of spatial cells at odds. It is therefore instructive to again examine how the phase-space elements evolve in classical mechanics. There, one may divide the classical phase-space into small cells of different sizes at will. Since each cell is frozen with the phase-space fluid, the cell volume is conserved at all time. By analogy, we should also let the spatial cell area θ^{12} be frozen with the incompressible flow,

$$\frac{d\theta^{12}}{dt} \equiv \partial_t \theta^{12} + v^i \partial_i \theta^{12} = 0. \tag{11}$$

This condition is exactly what we have stressed earlier that (7) applies only in the local rest frame.

Equation (11) can be verified by from another perspective, by noting that the electric field vanishes ($B'_{0i} = 0$) in the local rest frame of the string endpoint. A global reference frame that sees the string endpoint moving at a velocity v^j should also see an induced electric field $E_i = B_{ij} v^j$. It then follows from the Faraday's law, $\partial_t \mathbf{B} = \nabla \times \mathbf{E}$, of (10) that in this global frame

$$\frac{dB}{dt} = 0. \tag{12}$$

It is the regime of a perfectly conducting fluid where the "magnetic" field is frozen with the fluid; so is θ in the decoupling limit and hence (11) follows.

Equation (12) provides a partial evidence for us to relate $B\hat{z}$ to the hydrodynamic vorticity $\nabla \times \mathbf{v}(\mathbf{x}, t) \equiv (\partial_1 v_2 - \partial_2 v_1)\hat{z}$, which satisfies the same equation of motion. However, there can be infinitely many functions satisfying the same classical frozen-in equation of motion. Hence one needs to seek a direct link of B to the hydrodynamic vorticity. The direct link will be shown to be provided by a pair of canonical-conjugate fields, the quantization rule of which can be turned into the wanted link.

Proceeding along this line requires the action of classical two-dimensional incompressible hydrodynamics to be identified. We will not specify the detailed nature of the weak coupling among open strings, as we are only interested in the very low-energy physics where the details of interactions are smeared away. When the interactions are turned on, string endpoints can move across the **B** field as an incompressible fluid. The incompressible Navior-Stokes equation reads:

$$\partial_t \mathbf{v} - \mathbf{v} \times \nabla \times \mathbf{v} = -\nabla\left(\frac{\mathbf{v}^2}{2} + P\right), \tag{13}$$

where P is the pressure, serving as a constraint to ensure the incompressibility. Here, and from now on, all vector products and vector differentiations refer to the closed-string metric $g_{ij} = \epsilon^2 \delta_{ij}$ and $g_{00} = -1$.

Taking a divergence on (13), we see P satisfy an instantaneous Poisson equation, so that the scalar P is a nonlinear and non-local function of the pseudo-vector velocity field $\mathbf{v}$. The pressure does play a crucial role in the dynamics; for example, though not directly appearing in the energy density $T^{00} = \mathbf{v}^2/2$, the pressure contributes to the energy flux $T^{0i} = [v^i((\mathbf{v}^2/2) + P)]$ in a crucial way. The pressure indicated here is the dynamical pressure and not the actual pressure of the system; the latter is much greater than the former. In the context of quantum mechanics, the actual pressure is provided by the uncertainty principle. For example, writing the wave function of the Schroedinger equation as $f \exp(iS/\hbar)$, the real and imaginary parts of this complex equation are the conservation of probability and conservation of momentum respectively. In the momentum equation, the uncertainty-principle pressure arises from the second derivative of f, i.e., $-f\nabla^2 f/2$ [16], which can be much greater than the kinetic energy density $f^2(\nabla S)^2/2$ when ∇S means to describe the low-energy effective motion. The non-locality and nonlinearity of P as well as the appearance of different parities in this fluid problem suggest that incompressible hydrodynamics involves a more sophisticated action-principle formulation than the potential-flow hydrodynamics, which is basically a problem of a single complex-scalar-field [16].

In seeking the action of incompressible hydrodynamics, we begin with the local conservation laws in an ideal fluid that contains infinitely many local invariants, all satisfying the frozen-in condition. One may choose two of these invariants to be the new endpoint coordinate (α, β). This new coordinate is similar to the Lagrangian coordinate in fluid mechanics, the volume element of which, $|\nabla\alpha \times \nabla\beta| d\mathbf{r}^2$, is also a local invariant.

The complication of this hydrodynamic problem arises from the incompressibility condition, $\nabla \cdot \mathbf{v} = 0$, which must be explicitly reinforced at all time. Unlike the phase-space flow, where the incompressibility condition is automatically built in by its symplectic structure, here we need to impose a constraint to maintain the incompressibility.

The following Lagrangian density describes the evolution of the Lagrangian coordinates α and β with a built-in incompressibility condition:

$$L = -\epsilon^2 [\alpha \partial_t \beta + \frac{(\nabla\psi + \alpha\nabla\beta)^2}{2}], \tag{14}$$

where α, β and ψ are regarded as independent fields and the scaling factor ϵ^2 comes from $\sqrt{-Det(g_{\mu\nu})}$ in the Lagrnagian density. This Lagrangian is obtained by taking the non-relativistic and stiff-equation-of-state limits of the relativistic hydrodynamic Lagrangian [17]. Variations of this Lagrangian with respect to β, α and ψ yield

$$\frac{d\alpha}{dt} = 0, \qquad \frac{d\beta}{dt} = 0, \qquad \nabla \cdot \mathbf{v} = 0, \tag{15}$$

where

$$\mathbf{v} \equiv \nabla\psi + \alpha\nabla\beta. \tag{16}$$

The vorticity $\omega\hat{z} \equiv \nabla \times \mathbf{v} = \nabla\alpha \times \nabla\beta$. (It turns out that (14), (15) and (16) are also valid for three-dimensional incompressible hydrodynamics.) As both constant-α and constant-β lines are frozen, their intersections, which can be regarded as point vortices, are also frozen with the flow. Moreover, since ω is the density of vortices and the flow is incompressible, it thus follows that

$$\frac{d\omega}{dt} = \frac{d}{dt}(\hat{z} \cdot \nabla\alpha \times \nabla\beta) = 0. \tag{17}$$

This equation is nothing more than to say that the Jacobian between the effective "Lagrangian" coordinate and the Eulerian coordinate is a local invariant. In fact from the above frozen-in picture, the second equality of (17) holds for any incompressible velocity field $\mathbf{v}$, even when they are unrelated to α and β. Verification of (17) requires some algebra. A systematic way to show (17) is to decompose the strain tensor $\partial v^i/\partial x^j$ into a traceless component and an anti-symmetric component, and then make use of the frozen-in conditions for α and β to derive (17).

We proceed to examine what the constraint field ψ means. Construct the energy flux T^{0i} from (14) and compare it with the T^{0i} given four paragraphs earlier. One finds that

$$\frac{d\psi}{dt} = \frac{\mathbf{v}^2}{2} - P. \tag{18}$$

Thus ψ accounts, in part, for the accumulated effects of the non-local pressure P over the past evolution.

The Lagrangian (14) further allows us to identify the conjugate momentum of β,

$$\pi_\beta = -\epsilon^2\alpha. \tag{19}$$

The fields $-\epsilon^2\alpha$ and β are the canonical conjugate pair. Upon applying the quantization rule, they satisfy

$$[\alpha, \beta] = i\hbar\epsilon^{-2}. \tag{20}$$

Equation (20) is the desired condition that provides the direct link between the magnetic field and the vorticity.

The physical meanings of β and α can be best illustrated by the stationary flows. A two-dimensional stationary flow can be constructed using an auxiliary scalar field χ, where

$\mathbf{v} = \hat{z} \times \nabla\chi$ and the vorticity $\omega = \nabla^2\chi$. Since $\mathbf{v} \cdot \nabla\omega = 0$ in a stationary flow, it follows that $\nabla^2\chi = U(\chi)$ for an arbitrary function U. Thus, there exist infinitely many stationary-flow solutions, $\chi(x, y)$. The simplest stationary flows are the planar flows and circular flows, where the velocity is along the direction invariant to translation and rotation, i.e., the (angular) velocity being a Killing vector. In these two special cases, the stationary flow profile can be arbitrary.

A planar shear flow $\mathbf{v} = \hat{y}V(x)$ has no pressure gradient and it yields $\beta = y - V(x)t$, $\alpha = V(x)$ and $\psi = t(V(x)^2/2)$. A circular shear flow $\mathbf{v} = \hat{\phi}r\Omega(r)$, on the other hand, gives $\beta = \phi - \Omega(r)t$, $\alpha = r^2\Omega(r)$ and $\psi = t[(r^2\Omega^2/2) - \int r\Omega^2 dr]$. It is clear that β is nothing but the co-moving coordinate in the direction of the flow; the flow speed (α) is the negative conjugate momentum. We shall, from now on, call the canonical coordinate (α, β) to be the Lagrangian coordinate for convenience, though the Lagrangian coordinate in fluid mechanics has a slightly different meaning.

We are now ready to relate ω to B, with $2\pi B^{-1} = \theta^{12}$. Expressing $\alpha = \alpha(x)$, $\beta = y - t\alpha(x)$ for the planar shear flow, we find from (7) and (20) that

$$[\alpha, \beta] = \frac{d\alpha}{dx}[x, y] = i2\pi\omega B^{-1} = i\hbar\epsilon^{-2}, \tag{21}$$

where the commutator is defined in terms of the star products (8), and we have let $\theta^{ij} = \theta\epsilon^{ij}$. The first equality of (21) is valid only up to $O(\theta^2)$. Note from the third equality of (21), if both ω and B assume classical values, the small parameter ϵ will be of quantum origin, as it should scale as $\sqrt{\hbar}$.

One may, in fact, do better than the $O(\theta^2)$ order for (21), where all high-order terms vanish. This can be shown by changing the Cartesian coordinate $\mathbf{x} = (x, y)$ to the Lagrangian coordinate $\eta = (\alpha, \beta)$. With the latter coordinate, the new noncommutative parameter becomes $\hbar\epsilon^{-2}$. The star product for the Lagrangian coordinate assumes the same conventional form:

$$f(\eta) * h(\eta) = \exp[i(\hbar/2\epsilon^2)\epsilon^{ij}\partial_i^a\partial_j^b] f(\eta_a)h(\eta_b)|_{\eta_a=\eta_b=\eta}. \tag{22}$$

We now have $x = x(\alpha)$ and $y = \beta + t\alpha$. A straightforward calculation shows that $[x, y]$ is terminated beyond $O(\hbar)$ in the series expansion of the star product (22), and $[x, y] = i\hbar\epsilon^{-2}\omega^{-1}$. By definition, we also have $[x, y] = 2\pi i B^{-1}$. Hence (21) is valid for all orders of θ, and the vorticity ω can be identified to be $\hbar B/2\pi\epsilon^2$ in a planar flow.

A similar result also holds for the circular shear flow. Originally,

$$[\alpha, \beta] = i\theta\omega = i\hbar\epsilon^{-2} \tag{23}$$

up to $O(\theta^2)$. One now changes the polar coordinate $(r^2/2, \phi)$ to again the Lagrangian coordinate $\eta = (\alpha, \beta)$, where the coordinate transformation reads $r^2 = r^2(\alpha)$ and $\phi = \beta + t[\alpha/r^2(\alpha)]$. It follows that $[r^2/2, \phi] = i\hbar(rdr/d\alpha) = i\hbar\epsilon^{-2}\omega^{-1}$, valid for all orders of $\hbar$. By definition $[r^2/2, \phi] = i\theta$. Hence (23) is also valid for all orders of θ, and again $\omega = \hbar B/2\pi\epsilon^2$ in a circular flow.

In the above, we have used specific examples to identify ω with the background field B. However, the fact that $\omega = \hbar B/2\pi\epsilon^2$ to all orders holds only for a very specific coordinate for a given $\theta(x, y)$. In general this relation is valid only at the Poisson's level, i.e., to the lowest-order star-product expansion. Due to the existence of a θ-gradient, it makes the commutator between the two space components of one set of coordinate, e.g, $[x, y]$, different from that of another set, e.g., $[\bar{x}, \bar{y}]$, by an amount $O(\hbar^3)$. These high-order quantum effects break the area-preservation condition for different sets of coordinates, which are classically related by area-preserving transformations. This feature is new and generated by the θ-gradient. It is explored below.

III. QUANTUM AREA-PRESERVING MAPS AND SOLITONS

Area-preserving maps connect an equivalent class of coordinates, in the sense that a function expressed in terms of one set of coordinate possesses the same properties as that in terms of another set of coordinate of the equivalent class. In the commutative space, the area-preserving map is a transformation that maps every local point in the old frame to a new frame in an area preserving manner. However, in the noncommutative space, there no longer exist local points but unit cells of finite areas, as a result of the space quantum fluctuations. Thus, the quantum area preservation involves non-local conditions when measured in the classical space. When all unit cells have the same areas, i.e., a constant θ, the quantum area-preserving maps can still retain those good properties of the classical area-preserving maps. To be specific, the coordinates are operators in the noncommutative space and a well-defined function of space coordinates in the commutative space can become ill-defined in the noncommutative space, because a function of operator coordinate depends also on the ordering of operators, a feature that reflects the fuzziness of the noncommutative space. However, once a function is well-defined with one set of operator coordinate, it will remain so with another set of operator coordinate, provided that these coordinates are connected by an area-preserving transformation. The canonical coordinates in conventional quantum mechanics provide an example of such. However, when the unit cells have various sizes, their quantization can be a subtle problem [18]. It turns out that the Lagrangian coordinate can help solve this problem.

The examples given in the last section have demonstrated the usefulness of the Lagrangian coordinates, which turn complicated star-product calculations with a non-uniform θ into the simpler one with a constant noncommutative parameter $\hbar\epsilon^{-2}$. The Lagrangian coordinates also allow for construction of quantum area-preserving, or canonical, transformations in a systematic way. A canonical transformation transforms the canonical coordinate (α, β) to a new canonical coordinate $(\bar{\alpha}, \bar{\beta})$, such that $[\bar{\alpha}, \bar{\beta}]$ remains to be $i\hbar\epsilon^{-2}$. Much like in the commutative space, a finite quantum canonical transformation can be generated by successive applications of many infinitesimal canonical transformations. Infinitesimal transformations of the form, $\bar{\alpha} = \alpha - \partial\delta S/\partial\beta$ and $\bar{\beta} = \beta + \partial\delta S/\partial\alpha$, are canonical, where δS is the generation function, an analytical function of α and β. One can easily show that $[\alpha, \delta\beta] + [\delta\alpha, \beta] = 0$, so that they indeed are canonical.

Though being useful for its the symplectic structures, the Lagrangian coordinate is a deformed coordinate after all. Moreover, the Lagrangian coordinate describes the non-inertial frame that accelerates/decelerates with the fluid elements, thereby yielding a time-dependent metric $g'_{ij}(\alpha, \beta, t)$ and $g'_{i0}(\alpha, \beta, t)$, even in the presence of a stationary flow. Therefore, to understand the space quantum fluctuations, it is more relevant to examine the quantum fluctuations of the inertial-frame coordinate, described by the static metric $g_{ij} = \epsilon^2 \delta_{ij}$, than those of the Lagrangian coordinate.

In the noncommutative space, the static coordinates, such as the Cartesian coordinate, can be canonical coordinates when θ^{ij} is uniform, and hence different canonical coordinates can be generated by area-preserving maps. A function expressed in terms of these different canonical coordinates is equally well-defined. However, when θ^{ij} is non-uniform, all inertial-frame coordinates become non-canonical, i.e., the commutator not being a constant. As a consequence, quantum area-preserving maps among these inertial-frame coordinates generally do not exist, resulting in that functions expressed in terms of the inertial-frame coordinates become coordinate-dependent.

This peculiar feature implies that different inertial-frame coordinates suffer from quantum fluctuations in different ways, and hence they become not equivalent to each other. From this observation, it now becomes clear that area preservation imposes a subtle condition to ensure the quantum fluctuations of coordinates to be transformed in the same manner as the classical coordinates. That is, the Lagrangian coordinate can package all high-order quantum effects of star product in a neat manner, as if the quantum fluctuations were non-existent.

Despite such an unpleasant feature of the inertial-frame coordinates, it turns out that there exists a class of non-uniform θ^{ij}, or vortex flows, that can make the high-order quantum fluctuations in the inertial-frame coordinates vanish, and therefore area-perserving maps can be restored. These special flows are classical objects that survive in the noncommutative space. Such vortex flows will be called the "soliton" solutions and shown to be the solutions of the Dirac-Born-Infeld Lagrangian [18]. Below, we explore them.

Among all possible choices of inertial-frame coordinates, the holonomic and orthogonal coordinates are suited for describing the global space support of a function, and hence can naturally be extended to the operator coordinates in the noncommutative space. The most natural holonomic and orthogonal inertial-frame coordinate is the Cartesian coordinate. In the two-dimensional *commutative* space, the Cartesian coordinate can be transformed to one and only one holonomic and orthogonal coordinate in an area-preserving manner; it is the special polar coordinate $((r^2+r_0^2)/2, \phi+\phi_0)$, where r_0^2 and ϕ_0 are constants. One may prove this statement by showing: (a) one component, e.g., f, of the transformed coordinate must have a dimension, $(\text{length})^2$, and the other component, e.g., h, a dimension, $(\text{length})^0$; (b) with $f = r^2\tilde{f}(\phi)$ and $h = h(\phi)$, the desired result follows.

We shall therefore be confined to examining under what conditions the quantum area-preserving transformation between the special polar and Cartesian coordinates holds in the noncommutative space. A class of stationary flows with $\omega = \omega(r) = \hbar\epsilon^{-2}B(r)/2\pi$ will be examined in details. We have the commutator of the special polar coordinate, $[(r^2+r_0^2)/2, \phi+\phi_0] = [r^2/2, \phi]$, and it has been shown earlier that $[r^2/2, \phi] = i\hbar\epsilon^{-2}\omega^{-1}(r)$ to all orders of $\hbar$. Let θ be defined by (7), we next explore how to make θ equal to $\hbar\epsilon^{-2}\omega^{-1}$.

A convenient basis for evaluating $[x, y]$ is again the Lagrangian coordinate, $\eta = (\alpha, \beta)$, where x and y are written as

$$x(\eta_a) = r(\alpha_a)\cos[\beta_a + \frac{t\alpha_a}{r^2(\alpha_a)}], \quad y(\eta_b) = r(\alpha_b)\sin[\beta_b + \frac{t\alpha_b}{r^2(\alpha_b)}]. \tag{24}$$

Note that $x_a y_b = r(\alpha_a)r(\alpha_b)\cos(q_a)\sin(q_b) \propto (1/2)(\sin(q_+) + \sin(q_-))$, where $q_+ \equiv q_a + q_b$ and $q_- \equiv q_a - q_b$, with $q \equiv \beta + t\alpha/r^2(\alpha)$. Hence $x_a y_b$ contains four terms proportional to $\exp(\pm iq_+)$ and $\exp(\pm iq_-)$. Moreover, the operator $\exp[i(\hbar\epsilon^{-2}/2)\epsilon^{ij})\partial_i^a\partial_j^b]$ of the star product in (22) can be re-expressed as $\exp[i\hbar\epsilon^{-2}(\partial_{\alpha_-}\partial_{\beta_+} - \partial_{\alpha_+}\partial_{\beta_-})]$. When x and y are substituted for f and h in (22), one may perform Fourier transformations for β_+ and β_- and generates four Fourier modes for the ordinary product $x_a y_b$.

The star-product for these modes has the form

$$e^{\pm[i\beta_- + \hbar\epsilon^{-2}(\partial/\partial\alpha_+)]}F(\alpha_a, \alpha_b) + e^{\pm[i\beta_+ - \hbar\epsilon^{-2}(\partial/\partial\alpha_-)]}H(\alpha_a, \alpha_b). \tag{25}$$

The operator, $\exp[\pm\hbar\epsilon^{-2}(\partial/\partial\alpha_+)]$, is simply a shift operator that shifts α_+ in F by $\pm\hbar\epsilon^{-2}$. Likewise, $\exp[\mp\hbar\epsilon^{-2}(\partial/\partial\alpha_-)]$ shifts α_- in H by $\mp\hbar\epsilon^{-2}$. It thus follows

$$x * y = \frac{i}{4}[r^2(\alpha + \frac{\hbar}{2\epsilon^2}) - r^2(\alpha - \frac{\hbar}{2\epsilon^2})]$$
$$-\frac{1}{2}r(\alpha + \frac{\hbar}{2\epsilon^2})r(\alpha - \frac{\hbar}{2\epsilon^2})\sin[2\beta + t(\frac{2\alpha + \hbar/\epsilon^2}{2r^2(\alpha + (\hbar/2\epsilon^2))} + \frac{2\alpha - \hbar/\epsilon^2}{2r^2(\alpha - (\hbar/2\epsilon^2))})], \quad (26)$$

and hence

$$[x, y] = \frac{i}{2}[r^2(\alpha + \frac{\hbar}{2\epsilon^2}) - r^2(\alpha - \frac{\hbar}{2\epsilon^2})]. \quad (27)$$

Apparently, $[x, y] \neq [r^2/2, \phi]$. If we define $[x, y]$ to be $i2\pi B^{-1}$, c.f., (7), then the background field B is not related to the vorticity $\omega (= 2(dr^2/d\alpha)^{-1})$ by $\hbar B/2\pi\epsilon^2$, but by the non-local condition given on the right-hand side of (27). At the Poisson level, i.e., to the leading order in $\hbar$, we indeed have $i2\pi B^{-1} = [x, y] \approx (i\hbar/2\epsilon^2)dr^2/d\alpha = i\hbar\epsilon^{-2}\omega^{-1} = [r^2/2, \phi]$, and hence $\omega \approx \hbar B/2\pi\epsilon^2$. The deviation from canonical condition between the polar and Cartesian coordinates, or from $\omega = \hbar B/2\pi\epsilon^2$, starts from $O(\hbar^3)$ and only odd powers of $\hbar$ contribute to the difference. The deviation is of quantum origin since it involves $\hbar$.

The soliton solutions are those classical vortex flows that can survive in the noncommutative space by making $[x, y] = [r^2/2, \phi]$, or equivalently making $\omega = \hbar B/2\pi\epsilon^2$. These soliton solutions satisfy $[x, y] = (i\hbar/2\epsilon^2)dr^2/d\alpha$, thus obeying a linear differential-difference equation:

$$\hbar\epsilon^{-2}\frac{dr^2(\alpha)}{d\alpha} = r^2(\alpha + \frac{\hbar}{2\epsilon^2}) - r^2(\alpha - \frac{\hbar}{2\epsilon^2}). \quad (28)$$

The solution to this equation is not unique. Due to the linearity of (28), all solutions can be superposed. One exact solution that can be approximated by the increasingly higher-order expansion of small-$\hbar$ satisfies $d^n r^2/d\alpha^n = 0$ for $n \geq 3$. This solution terminates the $\hbar$-expansion and only the $O(\hbar)$ term survives It yields that $r^2 = c_2\alpha^2 + c_1\alpha + c_0$, or

$$\alpha = a\sqrt{1 \pm (r/r_0)^2} + b, \quad (29)$$

where r_0, a, b and c's are all constants. Though the Planck constant $\hbar$ appears in (28), it disappears in the solution (29), indicative of these soliton solutions to be of classical origin.

Note that α is the angular momentum, and a finite-angular momentum near $r = 0$ produces a singular flow $V_\phi \sim r^{-1}$. To avoid the singular behaviors, we may choose the plus sign in the square-root of (29) and let $b = -a$, thus obtaining

$$\alpha(r) = a(\sqrt{1 + (\frac{r}{r_0})^2} - 1), \quad \omega(r) = \frac{a}{r_0\sqrt{r_0^2 + r^2}}, \quad G_{ij}(r) = (\alpha'\epsilon)^2 \frac{a^2}{4\pi^2\hbar^2 r_0^2(r_0^2 + r^2)}, \tag{30}$$

where Eq.(1) in the decoupling limit has been used to obtain the open-string metric G_{ij}, given that $g_{ij} = \epsilon^2\delta_{ij}$. Note that if all flow quantities and the B-field are of classical values, we need ϵ^2 to scale as $\hbar$, thereby yielding a finite classical G_{ij} of order $\hbar^0$ and ϵ^0.

It turns out that the other choice of sign in the square-root of (29) can also make sense. Here we may alternatively have

$$\alpha(r) = a(\sqrt{1 - (\frac{r}{r_0})^2} - 1), \quad \omega(r) = \frac{-a}{r_0\sqrt{r_0^2 - r^2}}, \quad G_{ij}(r) = (\alpha'\epsilon)^2 \frac{a^2}{4\pi^2\hbar^2 r_0^2(r_0^2 - r^2)}. \tag{31}$$

This is a confined flow, where the vorticity diverges at the rotating boundary $r = r_0$.

Equation (30) shows a vortex solution that has a rigid-body rotation ($V_\phi \sim r$) at $r \ll r_0$ and exhibits a flat rotation ($V_\phi \to const.$) at $r \gg r_0$. Interestingly, in the limit $r_0 \to 0$, we have the open string metric $G_{ij} \propto r^{-2}$, making $G_{ij}dx^i dx^j = (d\ln r)^2 + d\phi^2$. Since ϕ is periodic and $-\infty < \ln r < \infty$, the topology of the open string metric becomes a cylinder. In other words, the open-string and closed-string metrics are both flat in this limit, but with different topologies as a result of the presence of a soliton vortex.

On the other hand, the open-string metric satisfies $G_{ij} \propto (r_0^2 - r^2)^{-1}$ for the other α given by (31). This soliton solution can have an interesting connection to magnetic monopoles. The open-string spatial line element can be expressed as: $dl^2 = r_0^2[d\xi^2 + \tan^2(\xi)d\phi^2]$, where $r \equiv r_0 \sin\xi$, and the non-relativistic kinetic energy is nothing but

$$T = \frac{r_0^2}{2}[\dot{\xi}^2 + \tan^2(\xi)\dot{\phi}^2] = \frac{r_0^2}{2}[\dot{\xi}^2 + \cot^2(\xi)p_\phi^2], \tag{32}$$

where p_ϕ is the ϕ-momentum. This form of kinetic energy may be derived from the metric of a symmetric rotor in three dimensions: $dl^2 = d\xi^2 + \sin^2(\xi)d\phi^2 + s^2(d\psi - \cos(\xi)d\phi)^2$, where s is a real constant. Again, express the non-relativistic kinetic energy from this line element. Upon recognizing the momenta $p_\psi = s^2(\dot\psi - \cos(\xi)\dot\phi)$ and $p_\phi = (\sin^2(\xi) + s^2\cos^2(\xi))\dot\phi - s^2\cos(\xi)\dot\psi$, we find that

$$T = \frac{1}{2}[\dot\xi^2 + \frac{(p_\phi + \cos(\xi)p_\psi)^2}{\sin^2(\xi)} + s^{-2}p_\psi^2]. \tag{33}$$

In the case $p_\psi \gg p_\phi$ or $p_\phi = 0$, (33) is reduced to (32) apart from a zero-level energy $s^{-2}p_\psi^2$, which can be made arbitrarily small when $s^2 \to \infty$. Note that the $p_\psi \gg p_\phi$ limit is reminiscent of the decoupling limit taken for the string metric, where the particle gyro-motion is negligible and the actual motion is governed by the diamagnetic current. Therefore, the dynamics in the open-string metric constructed by the vortex solution (31) is equivalent to the low orbital-angular-momentum dynamics of a three-dimensional rotor, for which the "spin"(ψ)-degree of freedom provides an effective background field. In fact, if we ignore the last term of (31) by taking $s^2 \to \infty$, the dynamics is identical to the one around a monopole of magnetic charge $Q_m(\propto p_\psi)$.

These solutions given in (30) and (31) are the classical ones, as all high-order terms of the star product vanish. Somewhat surprisingly, the vorticity ω found here turns out to be the electric-like classical gauge-field solutions, F_{0i}, of the Dirac-Born-Infeld (DBI) Lagrangian [9,19], $L \sim \sqrt{-Det(g_{\mu\nu} + \alpha' F_{\mu\nu})}$, created by a delta-function point source in the commutative space.

One may derive the Dirac-Born-Infeld solution straightforwardly by expressing $F_{0i} = -\partial_i\xi(r)$, where $d\xi/dr$ is to be identified as ω. Variation with respect to ξ yields the equation of motion:

$$\frac{d}{dr}[\frac{(\alpha')^2 r\xi'}{\sqrt{\epsilon^2 - (\alpha'\xi')^2}}] = 0, \tag{34}$$

where $\xi' \equiv d\xi/dr$. Taking the squared bracket to be a finite constant is equivalent to having a delta-function source at $r = 0$. Solutions (30) and (31) can be obtained from this procedure, and they correspond to having real and imaginary sources, respectively. In this decoupling limit, where $\alpha' \to \epsilon \to 0$, the DBI gauge-field solution typically has a finite amplitude on the order of $(\alpha')^0$, and a length scale of order unity when the source strength also scales as

ϵ. The $(\alpha')^0$ amplitude scaling is indeed what has been assumed for B or θ in the decoupling limit. Thus, the DBI solutions recover the correct scaling and are indeed valid classical solitons. We shall briefly comment on this point below.

These vortices studied here are the NS fields which obey the equation of motion (17) or $\omega = \nabla^2\chi = U(\chi)$ of an arbitrary function U in steady states, not compatible with the DBI gauge field that obeys a different equation of motion derived from the DBI Lagrangian. Only for some special choices of $U(\chi)$, the two give the same solutions, which survive the quantum fluctuations. In the present case, we have a rather complicated $U(\chi)$, for which the inverse function of $U(\chi)$ is $\chi = U^{-1}(\omega) = \omega^{-1} - (1/2)\ln(1 \pm \omega^{-1})$, where the " $+$ " and " $-$ " signs refer to solutions (30) and (31), respectively.

Another example that exhibits this feature is a magnetic-like DBI field, for which we need to represent $B_{12}(r) = d(rA_\phi)/d(r^2)$ in the DBI Lagrangian. Redefine $\bar{\xi} = rA_\phi(r)$, and perform the variation with respect to $\bar{\xi}$. With a point source, we find that B_{12} is a constant. Again when the source strength is of order ϵ, we recover a θ of correct ϵ scaling. This result recovers the familiar constant-θ case, where the high-order quantum fluctuations of the flat-space coordinates vanish. This constant ω solution corresponds to choosing a trivial $U(\chi) = const.$.

Indeed, it has been known that some classical DBI solutions can be ones that survive in the noncommutative space. They make the string world sheet conformal, whereby quantum corrections of high order vanish. The soliton solutions found here are also classical DBI solutions, and they make the two dimensional space in the D-brane world volume area preserving, thereby nullifying the space high-order quantum corrections. The connection between the two is not obvious since the string world sheet is outside the D-brane and involves two string endpoints, but the D-brane 2-sheet is within the brane. In addition, the area preservation and the space conformality are exclusive properties of coordinate transformations. Thus the two seem to be orthogonal to each other, and yet they give rise to similar results. There may be duality-like connections between those DBI solutions and the vortex solitons constructed by our way, but the connections are unclear at the moment.

IV. 2D DYNAMICS OF A STRONGLY MAGNETIZED 3D ELECTRON GAS

One of the known classical three-dimensional systems that exhibit two-dimensional incompressible hydrodynamic behaviors is the strongly magnetized three-dimensional electron gas [15]. The system is moderately long in the direction of magnetic field, with the electrons distributed uniformly along the magnetic field. The gyro-radius of electron is so small that only the guiding-center motion is relevant for the large-scale, low-frequency motion across the magnetic field. When the electron density is high, even a small amount of charge inhomogeneity is sufficient to produce a sizable electric field, which gives rise to electron drift motion perpendicular to the magnetic field with a velocity:

$$\mathbf{v} = c\frac{\mathbf{E} \times \mathbf{B}_0}{B_0^2}, \tag{35}$$

where $\mathbf{E}$ is the electrostatic electric field $-\nabla\Phi$, and $\mathbf{B}_0$ is the strong and uniform background magnetic field.

Take a divergence over (35), it shows that the velocity satisfies the incompressibility condition, $\nabla \cdot \mathbf{v} = 0$. On the other hand, the vorticity

$$\omega = \hat{z} \cdot \nabla \times \mathbf{v} = c\frac{\nabla^2\Phi}{B_0}, \tag{36}$$

and is proportional to the electron density excess/depletion $\delta\rho_e$. The vorticity ω has the same sign as the gyro-rotation for electron density excess, but opposite for density depletion (electron holes).

On the other hand, with incompressibility, the electron density excess/depletion $\delta\rho_e$ obeys the frozen-in condition. Therefore, from (36) the vorticity ω also obeys the frozen-in condition. It means that the collective equation of motion for these strongly magnetized electrons is the vorticity equation (17), or equivalently the Navior-Stokes equation. The example illustrates that in this coupled many-body system, the low-energy effective theory is the Navior-Stokes theory and not the original Newton-Maxwell theory. The electron system behaves like a neutral fluid is also evidenced by a comparison of the electric and kinetic energy densities. The former is $E^2/8\pi$ and the latter $n_e m_e v^2/2 = (E^2/8\pi)(4\pi n_e m_e c^2/B^2) \gg E^2/8\pi$ for a dense electron system in a very strong magnetic field, so strong that the magnetic energy density $B^2/8\pi$ is much larger than above two. Notice that no collisions, but small long-ranged electric forces, are needed to make electrons behave like an incompressible fluid. This is because the strong magnetic field has provided large pressure to yield a very stiff electron equation of state.

The electron gas exhibits quantum effects only at low temperature, and a cooled semiconductor in a strong magnetic field may be made to produce the quantum vortices studied here. This electron-gas system is similar to the one that shows quantum Hall effects, except that the latter has a small dimension along the magnetic field, i.e., monolayer systems, whereas the present one needs a finite longitudinal dimension. Although both systems exhibit two-dimensional dynamics, there is an important difference. The transverse electric-field, which is essential to produce vortex motion, is long-ranged for a charge colomn, but becomes short-ranged for a charge layer. The difference is reflected by the fact that the electric fields are mostly perpendicular both to the charge column and to the charge layer; the former lies perpendicular to the magnetic field but the latter aligns with the magnetic field, and therefore the spatial dependence of their transverse electric fields is different.

To make a comparison with what has been presented above, we note that given a strong background B_0 field, the change of magnetic field δB is negligible, so as not to over-pressure the electron gas for the slow vortex motion. Typically $\delta B/B_0$ is about the ratio of the electron gyro-period to the vortex rotation period, and hence in this strong-field regime, it is the vorticity, or the electron density perturbation $\delta\rho_e$ (c.f.,(36)), that plays the role of θ^{-1} in our previous discussions.

As discussed in Secs.(3) and (4), the Lagrangian coordinate (α, β) is useful for incorporating all quantum effects in the many-body system through the star product. Elemental vortices can be excited in this electron-gas column, and they are quantum objects different from the classical solitons given in (30) and (31). The elemental excitations tend to have angular momenta on the order of $\hbar$ but the classical solitons have an angular momentum of many $\hbar$. The fact that static inertial-frame coordinates are generally non-canonical, due to the high-order quantum corrections, may produce interesting experimentally detectable features. However in the following discussions, we shall make no attempt to examine the detailed quantum effects, but simply to predict a leading-order feature of these quantum vortices. Much like in the classical system, the quantum vortex is also charged; its charge can be quantized, but in a way different from that arising from the quantum Hall effect.

From (36), it follows that quantization of angular momentom, $L = m_e \int d^2\mathbf{r}\omega$, can be made equivalent to quantization of electric charge per unit length, $Q_e/l_\parallel = \int d^2\mathbf{r}\nabla \cdot \mathbf{E}/4\pi$, along the background field $\mathbf{B}_0$, and they are related by

$$Q_e/e = \frac{\pm 137n}{2}(\frac{l_\parallel \nu_{cy}}{c}), \tag{37}$$

where $l_\parallel$ is the system dimension along the magnetic field, $\nu_{cy} = eB_0/2\pi m_e c$ the electron cyclotron frequency with m_e being the electron mass, and the positive integer $n \equiv \pm L/\hbar$.

A estimate of relevant parameters can be in order. For a magnetic field $B_0 = 10^5$Gauss and the system size $l_\parallel = 10^{-3}$cm, we have $Q_e/e \approx \pm 1.37n/2$. Since the amount of charge is contributed by all electrons along the vortex coloumn of size $l_\parallel$, this length must be smaller than or comparable to the electron coherent length. At a sub-Kelvin temperature, the electron coherent length can be as large as 10^{-3}cm in a quality semiconductor sample, and such a quantum-vortex system can be made to exist.

For elemental vortex excitations, n is on the order of unity, the field strength B_0 and longitudinal system size $l_\parallel$ given above yield that the vortex quasi-particle contains a charge $Q_e \sim e$, which can even be a fraction of e. Notice that the elemental vortex excitations are expected to be extended objects, covering a transverse size much greater than both electron mean separation and electron gyro-radius $r_e(\equiv \sqrt{\hbar c/eB_0})$ at the first Laudau level. For the vortex dynamics to be close to two-dimensional, it also needs $l_\parallel$ still much greater than the transverse vortex size. With the field strength $B_0 = 10^5$Gauss, we have the electron gyro-radius $r_e \sim 10^{-6}$cm at the first Landau level, which is indeed much less than the longitudinal system size $l_\parallel$. The vortex transverse size thus lies in between 10^{-6} and 10^{-3}cm on the sub-micron to micron scales.

With this prediction, we are ready to compare this column electron-gas system with the layer electron-gas system that shows quantum Hall effects. Due to the short-range nature of the transverse electric field, the layer electron gas does not behave like a neutral fluid as the column electron gas does. Susskind investigated quantum Hall effects in connection with the noncommutative space by formulating the compressional displacement as the relevant gauge field [20], in contrast to the vorticity proposed in this work. The elemental excitation in the quantum Hall system is the charge hole, which is created, in Susskind's formulation, by strongly rarefying the local electron density, a mechanism to be contrasted with the present case where the electron density perturbation in a vortex is of small amplitude and extends over a larger area. Over one angular momentum quanta, Susskind's charge hole contains an integer fraction of electron charge, whereas in the present case we have an electric charge quantized directly not to the electron charge and having a dependence on the system length along the magnetic field.

V. DISCUSSIONS AND CONCLUSIONS

It is worthwhile to reiterate that incompressible hydrodynamics in the noncommutative space is a low-energy effective theory, where the the background B field is replaced by the vorticity field. The appearance of vortices can be analogous to the appearance of a new gauge field in an effective field theory. In fact, other than vortices classical hydrodynamics also supports pressure waves, i.e., the sound waves, that exist in the linear regime and are of high frequency. These high-frequency degrees of freedom are supported by the thermal pressure (or magnetic pressure for the case considered in Sec.(IV)), which is much greater than the nonlinear dynamical pressure P addressed in Sec.(III). These degrees of freedom have been filtered out (or integrated out in the semantics of field theory) in incompressible hydrodynamics, which deals only with low-frequency vortices. In many respects, the classical incompressible hydrodynamics indeed resembles the effective gauge-field theory of a coupled many-body system. High-energy excitations in quantum-field theories are the counterpart of sound waves, and upon integrating out high-energy contributions, the low-energy effective field is a new gauge field. Such an effective gauge field corresponds to the low-frequency vortices studied here.

In connection with the above general picture, we have also examined the strongly magnetized electron gas column in Sec.(IV). This electron system indeed exhibits two-dimensional incompressible vortex motion, due partly to the presence of a strong background magnetic field, which provides strong pressure to produce a stiff equation of state, and partly to the long-ranged, mean-field Coulomb interactions, which give rise to the $E \times B$ drift motion.

Though we have addressed various aspects of vortices, in the context of the string endpoint dynamics in a D3-brane, the much more delicate issue as to how these vortices are excited has not been touched upon. This aspect of problem can nevertheless be qualitatively understood from the classical physics as well. Vortices are intrinsically nonlinear objects, absent in the linear regime of hydrodynamics when there exists no background rotational flow. They can initially be excited through nonlinear sound interactions. However, sound waves are scalar fields but vortices consist of pseudo-vector fields. The excitation of the latter requires breaking of local chiral symmetry but with the global chiral symmetry to remain intact. Hence, vortices must be excited at least in pairs of opposite chirality. When vortices become densely populated in the fluid system, annihilation and formation of vortices through vortex-vortex interactions become the dominant processes. Moreover, vortices tend to merge to form larger ones. In a relaxed system, all small vortices tend to merge into a single big vortex, a manifestation of the Bose-Einstein condensation. Though the above description portrays the vortices of classical hydrodynamics, there is no reason to believe the quantum vortices should qualitatively behave differently. In particular, the soliton solutions given in (30) and (31) can be the Bose-Einstein condensates in relaxed systems.

In sum, we have shown in this paper that the frozen-in, or Lagrangian, coordinate forms a pair of canonical conjugate fields, which can describe the open-string endpoint dynamics via two dimensional incompressible hydrodynamics. The vorticity of hydrodynamics is shown to replace the background NS-field to the lowest order of the noncommutative parameter θ when θ is non-uniform and time-dependent. Despite the complications arising from the gradient of the noncommutative parameter, we have shown that in the Lagrangian coordinate, θ is replaced by a new noncommutative parameter $\hbar$, the Planck constant, thereby restoring the noncommutative space to the constant-θ geometry.

We have also identified the classical vortex flows that survive in the noncommutative space, and they turn out to be the classical solutions to the Dirac-Born-Infeld Lagrangian living only on the D-brane. As some Dirac-Bohr-Infeld gauge field can indeed have a similar property in making all quantum corrections vanish, we suspect that the two have deep connections, a good understanding of which may shed lights on gauge-field theories in noncommutative geometry with a non-uniform θ.

Finally we have also made a prediction for the existence of, and the quantized charge contained in, the quantum column vortices. These quantum vortices can be present in a strongly magnetized electron gas in a semiconductor of finite thickness at a sub-Kelvin temperature.

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REFERENCES

[1] E. Witten, Nucl. Phys. **B 268**, 253, 1986

[2] M.R. Douglas and C. Hull, Report No. JHEP 9802:008, 1998, hep-th/9711165

[3] Y. K. E. Cheung and M. Krough, Nucl. Phys. **B 528** 185, 1998

[4] C. S. Chu and P.M. Ho, Nucl. Phys. **B 550**, 151, 1998; C.S. Chu and P.M. Ho, Nucl. Phys. **568**, 447, 2000

[5] V. Schomerus, Report No. JHEP 9906:030, 1999, hep-th/9903205

[6] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, Report No. JHEP 9902:016, 1999, hep-th/9810072; F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, hep-th/9906161

[7] D. Bigatti and L. Sussikind, hep-th/9908056

[8] J.M. Maldacena and J.G. Russo, hep-th/9908134

[9] N. Seiberg and E. Witten, hep-th/9908142

[10] M. Li and Y.S. Wu, Phys. Rev. Lett. **84**, 2084, 2000

[11] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. **B 163**, 123, 1985

[12] C.G. Callen, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. **B 288**, 525, 1987

[13] A. Abouelsaood, C.G. Callen, C.R. Nappi and S.A. Yost, Nucl. Phys. **B 280**, 599, 1987

[14] M. Kontsevich, q-alg/9709040.

[15] T.B. Mitchell, C.F. Driscoll and K.S. Fine, Phys. Rev. Lett. **71**, 1371, 1993; K.S. Fine, A.C. Case, W.G. Flynn and C.F. Driscoll, Phys. Rev. Lett. **75**, 3277, 1995

[16] T. Chiueh, Phys. Rev. **E 57**, 4150, 1998; T. Chiueh, Phys. Rev. **E 61**, 3823, 2000

[17] T. Chiueh, Phys. Rev. **E 49**, 1269, 1994

[18] P.M. Ho and Y.T. Yeh, Phys. Rev. Lett. **85**, 5523, 2000; P.M. Ho and S.P. Miao, Phys.

Rev.**D 64**, 126002, 2001

[19] A.A. Tseytlin, " Born-Infeld Action, Supersymmetry and String Theory" in the Yuri Golfand memorial volume, ed. M. Shifman, World Scientific (2000), hep-th/9908105

[20] L. Susskind, hep-th/0101029